

PROJECTIVELY INDUCED ROTATION INVARIANT KÄHLER METRICS

FILIPPO SALIS

ABSTRACT. We classify Kähler-Einstein manifolds admitting a Kähler immersion into a finite dimensional complex projective space endowed with the Fubini–Study metric, whose codimension is less or equal than 3 and whose metric is rotation invariant.

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1. INTRODUCTION

In the present paper we address the problem of studying holomorphic and isometric (i.e. Kähler) immersions of Kähler–Einstein (from now on KE) manifolds into the finite dimensional complex projective space \mathbb{CP}^N endowed with the Fubini–Study metric g_{FS} . In particular, we believe the validity of the following

Conjecture. *Every Kähler-Einstein manifold which admits a Kähler immersion into (\mathbb{CP}^N, g_{FS}) , with $N < \infty$, is an open subset of a compact homogeneous space.*

An explicit example of non-homogeneous KE manifolds which admit a Kähler immersion into $(\mathbb{CP}^\infty, g_{FS})$ can be found in [12].

The main result of the paper consists in proving the above mentioned conjecture in the case of *rotation invariant* Kähler metrics when the codimension with respect to the target projective space is ≤ 3 . This is shown in the next section via the following two theorems.

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Theorem 1.1. *The Einstein constant λ of a KE rotational invariant and projectively induced n -dimensional manifold M^n is a positive rational number less or equal to $2(n+1)$. Hence, if M^n is complete then M^n is compact and simply connected.*

Theorem 1.2. *Let (M, g) be an n -dimensional KE manifold whose metric is rotation invariant. Then (M, g) admits a Kähler immersion into \mathbb{CP}^{n+k} for k less than or equal to 3, if and only if (M, g) is an open subset of (\mathbb{CP}^n, g_{FS}) , $(\mathbb{CP}^2, 2g_{FS})$ or $(\mathbb{CP}^1 \times \mathbb{CP}^1, g_{FS} \oplus g_{FS})$.*

Throughout the paper a Kähler metric g on a complex manifold M is *rotation invariant* if there exists a point $p \in M$, a coordinate system (z_1, \dots, z_n) centered in p and a Kähler potential Φ for g on a neighborhood of p such that Φ is rotation invariant in (z_1, \dots, z_n) , i.e. it only depends on $|z_1|^2, \dots, |z_n|^2$. Furthermore, a Kähler metric g on a connected complex manifold M will be called *projectively induced* if there exist a point $p \in M$, a neighbourhood V of p and a Kähler immersion $f: V \rightarrow \mathbb{CP}^N$, with $N < \infty$, i.e. $f^*g_{FS} = g|_V$. If M is projectively induced at p , then it is at any other point (cfr. [4, Theorem 10]). Moreover, in the case whether M is also simply connected then the Kähler immersion $f: V \rightarrow \mathbb{CP}^N$ can be extended to a Kähler immersion of the whole M into \mathbb{CP}^N (cfr. [4, Theorem 11]).

The content of theorems 1.1, 1.2 and of the above mentioned conjecture should be compared with the following results:

Theorem A (D. Hulin [9]). *If a compact KE manifold is projectively induced then its Einstein constant is positive.*

Theorem B (S. S. Chern [5], K. Tsukada [18]). *Let (M, g) be a complete n -dimensional Kähler-Einstein manifold ($n \geq 2$). If (M, g) admits a Kähler immersion into \mathbb{CP}^{n+2} , then M is either totally geodesic or the complex quadric Q_n in \mathbb{CP}^{n+1} .*

2. PROOF OF THE MAIN RESULT

In order to prove theorems 1.1 and 1.2, it is useful to recall the definitions of Calabi's diastasis function and Bochner's coordinates. Let (M, g) be a Kähler manifold with Kähler potential Φ . If g (and hence Φ) is assumed to be real analytic, by duplicating the variables z and \bar{z} , Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighbourhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here \bar{M} denotes the manifold conjugated to M). Thus one can consider the power expansion of Φ around the origin with respect to z and \bar{z} and write it as

$$\Phi(z, \bar{z}) = \sum_{j,k=0}^{\infty} a_{jk} z^{m_j} \bar{z}^{m_k}, \quad (1)$$

where we arrange every n -tuple of nonnegative integers as a sequence $m_j = (m_{j,1}, \dots, m_{j,n})$ and order them as follows: $m_0 = (0, \dots, 0)$ and if $|m_j| = \sum_{\alpha=1}^n m_{j,\alpha}$, $|m_j| \leq |m_{j+1}|$ for all positive integer j . Moreover, z^{m_j} denotes the monomial in n variables $\prod_{\alpha=1}^n z_{\alpha}^{m_{j,\alpha}}$.

A Kähler potential is not unique, as it is defined up to an addition with the real part of a holomorphic function. The *diastasis function* D_0 for g is nothing but the Kähler potential around p such that each matrix (a_{jk}) defined according to equation (1) with respect to coordinates system centered in p , satisfies $a_{j0} = a_{0j} = 0$ for every nonnegative integer j .

For any real analytic Kähler manifold there exists a coordinates system, in a neighbourhood of each point, such that

$$D_0(z) = \sum_{\alpha=1}^n |z_{\alpha}|^2 + \psi_{2,2}, \quad (2)$$

where $\psi_{2,2}$ is a power series with degree ≥ 2 in both z and \bar{z} . These coordinates, uniquely determined up to unitary transformation (cfr. [3], [4]), are called the *Bochner's coordinates* (cfr. [3], [4], [8], [9], [14], [16]).

In order to prove theorem 1.1 we need the following two lemmata, the first one dealing with Bochner's coordinates and KE metrics and the second with Bochner's coordinates and projectively induced and rotation invariant metrics.

Lemma 2.1 (cfr. [1] and [9]). *A Kähler manifold (M, g) is Einstein if and only if by choosing Bochner's coordinates on a neighbourhood U of a point p of a Kähler manifold (M, g) whose diastasis on U is given by $D_0(z)$, it satisfies the Monge-Ampère Equation*

$$\det(g_{\alpha\bar{\beta}}) = e^{-\frac{\lambda}{2} D_0(z)}. \quad (3)$$

Proof. Recall that requiring a Kähler metric g on a complex manifold M to be Einstein, i.e. there exists $\lambda \in \mathbb{R}$ such that

$$\rho = \lambda \omega, \quad (4)$$

where $\rho = -i\partial\bar{\partial} \log \det(g_{\alpha\bar{\beta}})$ is the Ricci form associated to g , is locally equivalent to require that it satisfies the Monge-Ampère equation

$$\det(g_{\alpha\bar{\beta}}) = e^{-\frac{\lambda}{2}(\Phi + f + \bar{f})}.$$

Therefore, it is easy to check, once set Bochner's coordinates are set, that the expansion of $\det(g_{\alpha\bar{\beta}})$ in the (z, \bar{z}) -coordinates around the origin reads $\det(g_{\alpha\bar{\beta}}) = 1 + h(z, \bar{z})$, where $h(z, \bar{z})$ is a power series in z, \bar{z} which contains only mixed terms (i.e. of the form $z^j \bar{z}^k$, $j \neq 0, k \neq 0$). Further, also the expansion of $D_0(z)$, given in equation (2), contains only mixed terms, forcing $f + \bar{f}$ to be zero. \square

We recall that a holomorphic immersion $f : U \rightarrow \mathbb{CP}^N$ is *full* provided $f(U)$ is not contained in any \mathbb{CP}^k for $k < N$.

Lemma 2.2. *Let g be a projectively induced Kähler metric on a complex manifold (M, g) and let $f : V \rightarrow \mathbb{CP}^N$ be the holomorphic immersion such that $f^*g_{FS} = g$. Assume that f is full and g admits a diastasis D_0 on a neighbourhood U of a point $p \in M$, which is rotation invariant with respect to the Bochner's coordinates z_1, \dots, z_n around p . Then there exists an open neighbourhood W of p such that $D_0(z)$ can be written on W as*

$$D_0(z) = \log \left(1 + \sum_{j=1}^n |z_j|^2 + \sum_{j=n+1}^N a_j |z^{m_{h_j}}|^2 \right), \quad (5)$$

where $a_j > 0$ and $h_j \neq h_k$ for $j \neq k$.

Proof. Up to a unitary transformation of \mathbb{CP}^N and by shrinking V if necessary we can assume $f(p) = [1, 0, \dots, 0]$ and $f(V) \subset U_0 = \{Z_0 \neq 0\}$. Since the affine coordinates on U_0 are Bochner's coordinates for the Fubini–Study metric g_{FS} then, by [4, Theorem 7 p. 15], f can be written on $W = U \cap V$ as:

$$f : W \rightarrow \mathbb{C}^N, z = (z_1, \dots, z_n) \mapsto (z_1, \dots, z_n, f_{n+1}(z), \dots, f_N(z)),$$

where

$$f_j(z) = \sum_{k=n+1}^{\infty} \alpha_{jk} z^{m_k}, \quad j = n+1, \dots, N.$$

Since the diastasis function is hereditary (see [4, Prop. 6 p. 4]) and that of \mathbb{CP}^N around the point $[1, 0, \dots, 0]$ is given on U_0 by $\Phi(z) = \log(1 + \sum_{j=1}^N |z_j|^2)$, where $z_j = \frac{Z_j}{Z_0}$, one gets

$$D_0(z) = \log(1 + \sum_{j=1}^n |z_j|^2 + \sum_{j=n+1}^N |f_j(z)|^2).$$

The rotation invariance of $D_0(z)$ and the fact that f is full implies that the f_j 's are monomial of z of different degree and formula (5) follows. \square

To simplify the notation, from now on we write P_{z_α} for $\partial P / \partial z_\alpha$, $P_{\bar{z}_\beta}$ for $\partial P / \partial \bar{z}_\beta$, $P_{z_\alpha \bar{z}_\beta}$ for $\partial^2 P / \partial z_\alpha \partial \bar{z}_\beta$ and so on.

Proof of Theorem 1.1. By Lemma 2.2 we can assume that there exists $p \in M$ such that the diastasis around p can be written as $D_0(z) = \log(P)$ where P is a polynomial in the variables $|z_1|^2, \dots, |z_n|^2$. By the equality

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 D_0(z)}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{P P_{z_\alpha \bar{z}_\beta} - P_{z_\alpha} P_{\bar{z}_\beta}}{P^2},$$

we have

$$\det(g_{\alpha\bar{\beta}}) = \det\left(\frac{PP_{z_\alpha\bar{z}_\beta} - P_{z_\alpha}P_{\bar{z}_\beta}}{P^2}\right) = \frac{1}{P^{2n}} \det(PP_{z_\alpha\bar{z}_\beta} - P_{z_\alpha}P_{\bar{z}_\beta}).$$

Given a polynomial Q in the variables $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$ we denote by $\deg Q$ the total degree of Q with respect to all the variables $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$. Then

$$\deg \det(PP_{z_\alpha\bar{z}_\beta} - P_{z_\alpha}P_{\bar{z}_\beta}) \leq 2n \deg P - 2n.$$

On the other hand from Monge–Ampère Equation (3) we get

$$\deg \det[(PP_{z_\alpha\bar{z}_\beta} - P_{z_\alpha}P_{\bar{z}_\beta})] - 2n \deg P = -\frac{\lambda}{2} \deg P,$$

which forces $\frac{\lambda}{2} \geq \frac{2n}{\deg P} > 0$. Thus, if M is complete, by Bonnet’s Theorem, M is also compact. Then M is simply connected by a well-known theorem of Kobayashi [10]. The final upper bound $\lambda \leq 2(n+1)$ follows from the following D. Hulin’s result. \square

Lemma 2.3 (D. Hulin [8]). *Let (V, h) be a KE manifold which admits a Kähler immersion into \mathbb{CP}^N . Then it can be extended to a complete n -dimensional KE manifold (M, g) and the Einstein constant is a rational number. Further, let this immersion be full and let the Einstein constant $\lambda = 2\frac{p}{q}$ be positive, where p/q is irreducible, then $p \leq n+1$ and if $p = n+1$ (resp. $n = p = 2$) then $(M, g) = (\mathbb{CP}^n, g_{FS})$ (resp. $(M, g) = (\mathbb{CP}^1 \times \mathbb{CP}^1, g_{FS} \oplus g_{FS})$).*

In order to prove Theorem 1.2 we also need:

Lemma 2.4. *Let (M, g) be a complete n -dimensional rotation invariant KE manifold. If $n > 2k$, (M, g) admits a (local) Kähler immersion into \mathbb{CP}^{n+k} if and only if $(M, g) = (\mathbb{CP}^n, g_{FS})$.*

Proof. By Lemma 2.2 there exist a point $p \in M$ and local coordinates z_1, \dots, z_n around it such that the diastasis function $D_0(z)$ for g centered at p can be written as

$$\begin{aligned} D_0(z) = \log(1 + \sum_{j=1}^n |z_j|^2 + \sum_{j=1}^n a_j |z_j|^4 + \\ + \sum_{1 \leq j < k \leq n} b_{jk} |z_j|^2 |z_k|^2 + \psi_{3,3}), \end{aligned}$$

where $\psi_{3,3}$ is a (rotation invariant) polynomial of degree not less than three both in z and in \bar{z} . For $h = 1, \dots, n$, deriving with respect to z_h and \bar{z}_h both sides of the Monge–Ampère Equation (3) and evaluating at 0, i.e. by considering the n equalities

$$\frac{\partial^2}{\partial z_h \partial \bar{z}_h} \left(\det \left(\frac{\partial^2 D_0(z)}{\partial z_j \partial \bar{z}_k} \right) \right) \Big|_0 = \frac{\partial^2}{\partial z_h \partial \bar{z}_h} \left(e^{-\frac{\lambda}{2} D_0(z)} \right) \Big|_0, \quad (h = 1, \dots, n),$$

we get n equations of the form

$$4a_h + \sum_{\substack{k=1 \\ k \neq h}}^n b_{hk} - (n+1) = -\frac{\lambda}{2}, \quad (h = 1, \dots, n), \quad (6)$$

in which $b_{ij} = b_{ji}$. Thus, if the codimension of M into \mathbb{CP}^N is $k < n/2$ at least one of these equations is of the form $\lambda = 2(n+1)$, because the polynomial P consists of only $n+k+1$ monomials and then at most k of the variables $\{a_i, b_{ij}\}_{\substack{1 \leq i \leq n \\ i < j \leq n}}$ can be different from 0. Therefore the thesis follows by Lemma 2.3. \square

We are now in the position of proving Theorem 1.2.

Proof of Theorem 1.2. By virtue of Lemma 2.3, we can consider directly the case where (M, g) is a complete KE projectively induced and rotation invariant manifold. By Lemma 2.2 there exist a point $p \in M$ and local coordinates z_1, \dots, z_n such that the diastasis function $D_0(z)$ around p can be written as

$$D_0(z) = \log \left(1 + \sum_{j=1}^n |z_j|^2 + \sum_{j=1}^n a_j |z_j|^4 + \sum_{1 \leq j < k \leq n} b_{jk} |z_j|^2 |z_k|^2 + \sum_{j=1}^n c_j |z_j|^6 + \sum_{j,k=1(j \neq k)}^n d_{jk} |z_j|^2 |z_k|^4 + \psi_{4,4} \right),$$

where $\psi_{4,4}$ is a rotation invariant polynomial of degree not less than four both in z and in \bar{z} . If $k = 0$, we have $D_0(z) = \log(1 + \sum_{j=1}^n |z_j|^2)$ and $(M, g) = (\mathbb{CP}^n, g_{FS})$. The statement holds by Theorem B for those case where the codimension is equal to 1 or 2. Thus let $k = 3$. By Lemma 2.4, the statement is true for $n > 6$. Here we need to analyze the cases $n \leq 6$ separately.

Consider the case $n = 2$. As in the proof of Lemma 2.4, for $h = 1, 2$, by deriving with respect to z_h, \bar{z}_h both sides of the Monge–Ampère Equation (3), and evaluating at 0 we get the 2 equations

$$\begin{aligned} 4a_1 + b_{12} &= 3 - \frac{\lambda}{2}, \\ 4a_2 + b_{12} &= 3 - \frac{\lambda}{2}, \end{aligned}$$

from which follows immediately $a_1 = a_2$ and $b_{12} = 3 - \frac{\lambda}{2} - 4a_1$. If $a_1 = a_2 = b_{12} = 0$, we get $\lambda = 6$, and by Lemma 2.3 $(M, g) = (\mathbb{CP}^2, g_{FS})$. If $a_1 \neq 0, b_{12} \neq 0$, since the codimension is 3, all the other coefficients must vanish, thus by deriving both sides of the Monge–Ampère Equation (3) by $z_1, \bar{z}_1, z_2, \bar{z}_2$, and twice by z_1 and \bar{z}_1 , evaluating at zero, we get a system, whose unique acceptable solution is $b_{12} = 1/2, a_1 = 1/4$ and $\lambda = 3$, that is $(M, g) = (\mathbb{CP}^2, 2g_{FS})$. If $a_1 \neq 0$ but $b_{12} = 0$, by deriving again both sides of the Monge–Ampère Equation (3) by $z_1, \bar{z}_1, z_2, \bar{z}_2$ and evaluating

at zero and at $\lambda = 2(3 - 4a_1)$ we have $4a_1 + 4d_{12} + 4d_{21} = 0$, impossible, since $a_1 \neq 0$ and all the coefficients must be non negative. It remains to consider the case $b_{12} \neq 0$, $a_1 = a_2 = 0$. By deriving the Monge–Ampère Equation twice by z_j and twice by \bar{z}_j (for $j = 1, 2$), evaluating at zero and at $a_1 = a_2 = 0$, $b_{12} = 3 - \frac{1}{2}\lambda$, we get $d_{12} = 9c_1$ and $d_{21} = 9c_2$. Since the codimension is 3, only two of them can be different from zero. If they are all zero, we get $\lambda = 6$ and by Lemma 2.3 $(M, g) = (\mathbb{CP}^2, g_{FS})$ or $\lambda = 4$ and again by Lemma 2.3 $(M, g) = (\mathbb{CP}^1 \times \mathbb{CP}^1, g_{FS} \oplus g_{FS})$. Let us suppose that two of them are different from zero, for example $d_{12} \neq 0$. Then all the terms of higher order vanish, and taking the third order derivative we get again $\lambda = 6$ or $\lambda = 4$.

The case $n = 3$ is very similar to that one. The system given in the proof of Lemma 2.4 reads

$$\begin{aligned} 4a_1 + b_{12} + b_{13} &= 4 - \frac{\lambda}{2}, \\ 4a_2 + b_{12} + b_{23} &= 4 - \frac{\lambda}{2}, \\ 4a_3 + b_{13} + b_{23} &= 4 - \frac{\lambda}{2}. \end{aligned}$$

It is easy to see that only three cases do not reduce immediately to $(M, g) = (\mathbb{CP}^3, g_{FS})$, that is $a_1 = a_2 = a_3 \neq 0$, or $a_1 = b_{23} \neq 0$ (and all the symmetric to them), or $b_{12} = b_{23} = b_{13} \neq 0$. By taking the second order derivative of the Monge–Ampère Equation and evaluating at zero, it follows that these cases are incompatible.

If $n = 4$, it easy to see from the system of linear equation

$$\begin{aligned} 4a_1 + b_{12} + b_{13} + b_{14} &= 5 - \frac{\lambda}{2}, \\ 4a_2 + b_{12} + b_{23} + b_{24} &= 5 - \frac{\lambda}{2}, \\ 4a_3 + b_{13} + b_{23} + b_{34} &= 5 - \frac{\lambda}{2}, \\ 4a_4 + b_{14} + b_{24} + b_{34} &= 5 - \frac{\lambda}{2}, \end{aligned}$$

that, up to symmetries, only the following cases may occur: all the coefficients are equal to zero, $a_1 = a_2 = b_{34} \neq 0$ or $b_{12} = b_{34} \neq 0$. In the third case, without loss of generality, we suppose that $d_{23} = d_{32} = 0$. Therefore if the second or third case holds, by deriving both sides of the Monge–Ampère Equation with respect to z_2 , z_3 , \bar{z}_2 , \bar{z}_3 , evaluating at zero and considering the relation above, we get $b_{34} = 0$, and conclusion follows.

The cases $n = 5$ and $n = 6$ are very similar to that one. For $n = 5$, by the system of linear equation either the coefficients of the system are zero, or up to

symmetries $4a_1 = b_{23} = b_{45} \neq 0$. Deriving with respect to $z_2, z_4, \bar{z}_2, \bar{z}_4$ the Monge–Ampère Equation and evaluating at zero, one gets $b_{23} = 0$. For $n = 6$, from the system of linear equation one gets that either the coefficients are all zero or $b_{12} = b_{34} = b_{56} \neq 0$. By deriving with respect to $z_2, z_4, \bar{z}_2, \bar{z}_4$ and evaluating at zero, one gets $b_{34} = 0$, and we are done. \square

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DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI CAGLIARI, VIA OSPEDALE 72,
09124 CAGLIARI (ITALY)

E-mail address: `filippo.salis@gmail.com`